1) Birthday problem — out of k people, what is the probability of 2 people having same birthday $= 1 - \frac{365P_k}{365^k}$

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2) Matching problem out of n cards

$P(\text{No match}) = \sum_{i=0}^{n} (-1)^i \frac{1}{i!} \sim \frac{1}{e}$ as $n \to \infty$

$P(k \text{ matches}) = \frac{1}{k!} \sum_{i=0}^{n-k} (-1)^i \frac{1}{i!} = nC_k \cdot P(\text{first } k \text{ cards matching}) \left[1 - P(A_{k+1} \cup A_{k+2} \cdots A_n)\right]$

$= nC_k \cdot \frac{1}{nP_k} \sum_{i=0}^{n-k} (-1)^n \frac{1}{i}$

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3) Gambler's Ruin
A & B
play with p & q
of winning each round

$P(A \text{ win} / \text{start with } i \text{ money of total } N) = \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N}$ $\quad q \neq p$

$\frac{i}{N}$ $\quad q = p$

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4) Putnam - local maxima in permutation of n integers $= \frac{1}{2} \times 2 + \frac{1}{3} \times (n-2) = 1 + \frac{n-2}{3} = \frac{n+1}{3}$

↳ Indicator for each position → Extreme posi (2)
↳ Middle posi

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5) St petersburg Paradox $\Rightarrow T = 2^x \$$ for #Tosses until (& including Head)

$E(T) = \sum_{x=1}^{\infty} 2^x \left(\frac{1}{2}\right)^x = \infty$ | if money limited & toss limited to k $\Rightarrow E(T) = \sum_{x=1}^{k} 2^x \left(\frac{1}{2}\right)^x = k \$$

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6) Using poisson in bday problem → Approx probability for k matches in a group of n people

$N = nC_k$ → subsets of people → very large

p = Probablity of a given subset of k people all having same bday $= \frac{365}{365^k} = \frac{1}{365^{k-1}}$ → very small

Let T = # subset of k people having same bday $\Rightarrow 1 - P(T=0) = P(\text{atleast 1 occurence of } k \text{ people with same bday}) = m$

$T \sim \text{poisson}\left(\lambda = Np = \frac{nC_k}{365^{k-1}}\right)$

$P(T=0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda}$

$\Rightarrow \boxed{m = 1 - e^{-\frac{nC_k}{365^{k-1}}}}$

eg $k=3$ $\quad m = 1 - e^{-\frac{nC_3}{365^2}}$

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7) Coupon Collector — n Vintage toy types in a series
$T = RV$ = # Toys until full set in Series

$E(T) = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right) \sim n \log n$ for large n

$= \boxed{n \sum_{k=1}^{n} \frac{1}{k} \sim n \log n}$

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10) Laplace Rule of Succession → $X_1, X_2, X_3 \cdots X_n$ Bern(θ) iid $\quad P(S_n|\theta) \sim \text{Bino}(n, \theta)$

θ = unknown $\sim$ Uniform(0,1) → Bayesian Interpret $\quad \boxed{X_1 + X_2 + X_3 + \cdots X_n = S_n}$

$P(\theta/S_n) = \frac{f(\theta) \cdot P(S_n = k|\theta)}{\int_{\theta=0}^{1} f(\theta) \cdot P(S_n|\theta)}$ $\Rightarrow P(\theta/S_n) = c \cdot nC_k \, \theta^k (1-\theta)^{n-k}$

normalised Constant indep of θ

c = normalised constant

$c \int_{\theta=0}^{1} \theta^n d\theta = 1 \Rightarrow \frac{c\theta^{n+1}}{n+1}\Big|_0^1 = 1$

$\boxed{c = n+1}$

Special case when $S_n = k = n$ $\quad P(\theta|S_n = n) = c \cdot \theta^n$

$\boxed{P(\theta|S_n) = (n+1) \theta^n}$

$P(X_{n+1}|S_n) = E(\theta|S_n) = \int_0^1 (n+1)\theta^{n+1} d\theta = \boxed{\frac{n+1}{n+2}}$

Distribution	Mean	Variance	MGF	$E(X^k)$	Relation to other distribution
Bernoulli (p) $E(I) = p$ $P(I) = \begin{cases} p, & 1 \\ 1-p, & 0 \end{cases}$	p	pq	$pe^t + q$	p	$I_1 + I_2 + \dots I_n \sim$ Binomial(n,p) $I_1 \dots I_n$ iid Bernoulli p
Binomial (n,p) $P(X=k) = {}^nC_k\, p^k q^{n-k}$	np	npq	$(pe^t+q)^n$		$X \sim Bino(n,p)$; $Y \sim Bin(m,p)$ $n - X \sim Bino(n,q)$ $X+Y \sim Bin(n+m,p)$
Hypergeometric(w,b,n) $X \sim HG(p,q,n,N)$ $\frac{{}^wC_k\, {}^bC_{n-k}}{{}^{w+b}C_n}$ $k \in \{0,1,2,\dots n\}$, $0 \le k \le w$, $0 \le n-k \le b$	$np = \frac{nw}{w+b}$	$npq \frac{N-n}{N-1}$ $\frac{nwb}{(w+b)^2} \frac{N-n}{N-1}$			$HG \simeq Bino(n,p)$ $N \to \infty$ $\boxed{w = Np \quad w+b = N \quad b = Nq}$
$X \sim$ Geometric (p) $P(X=k) = q^{k-1}p$ $k=1,2\dots$	$\frac{1}{p}$	$\frac{q}{p^2}$	• Memoryless $P(X = t+k \mid X > t) = P(X=t)$ $E(X \mid X>t) = t + \frac{1}{p}$ $E(X - t \mid X>t) = \frac{1}{p}$		$Y = X-1$; $X \sim$ geometric $E(Y) = \frac{q}{p}$; $Var(Y) = \frac{q}{p^2}$
$X \sim$ Neg Bino(r,p) $P(X=n) = p^r q^n \binom{n+r-1}{r-1}$ # failures before r success	$\frac{rq}{p}$	$\frac{rq}{p^2}$	$X \sim geo(p)$ $Y_i = X_i - 1$ $Y = Y_1 + Y_2 + \dots Y_r \sim NegBi(r,p)$		$Y + r = \#$ trails $E(Y) + r = r\frac{q}{p} + r = r\left(\frac{q}{p}+1\right) = \boxed{\frac{r}{p}}$
Poisson Distribution $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$	$X \sim Pois(\lambda)$ $Y \sim Pois(\mu)$ $X+Y \sim Pois(\lambda+\mu)$	$X \sim$ Poisson(λt) ; $P(X=0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$ $T \sim exp(\lambda)$ $P(X=0) = P(T>t)$
Uniform(a,b) $f_u = \frac{1}{b-a}$ $a \le u \le b$ $F_u = \frac{x-a}{b-a}$ $a \le u \le b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$U_1 + U_2 \to$ $U_1 - U_2 \to$ $\lvert U_1 - U_2 \rvert \to$		let $X \sim F_X(x)$; $U \sim Uni(0,1)$ $F_X(x)$ — CDF then $F^{-1}(U) = X$ • $P(F^{-1}(U) < k) = P(U \le F(k)) = F(k)$ • $F_X(X) \sim$ Uniform $(0,1)$
Uniform $[a,b]$ $P(X=x) = \frac{1}{b-a+1}$ $P(X \le x) = \frac{x-a+1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{N^2-1}{12}$	$E(\lvert U_1 - U_2 \rvert) = 1/3$ $M = Max(U_1, U_2)$ $L = Min(U_1, U_2)$ $E(M) = 2/3$	$E(L) = 1/3$	$X_1 \sim N(\mu_1, \sigma_1^2)$; $X_2 \sim N(\mu_2, \sigma_2^2)$ $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ $X_1 - X_2 \sim N(\mu_1-\mu_2, \sigma_1^2+\sigma_2^2)$ $-X_1 \sim N(-\mu_1, \sigma_1^2)$
Normal Distribution (μ, σ^2) $\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$	μ	σ^2	$e^{t\mu + \frac{t^2}{2}\sigma^2}$	$E(Z^{2n}) = \frac{2n!}{2^n n!} = (2n-1)!!$ skip factors $E(Z^{odd}) = 0$	$\left.\begin{array}{l} Z_1 \sim N(0,1) \\ Z_2 \sim N(0,1) \end{array}\right\} \Rightarrow \begin{array}{l} Z_1 - Z_2 \sim N(0,2) \\ Z_1 - Z_2 = \sqrt{2} Z \end{array}$ $E(\lvert Z_1 - Z_2 \rvert) = \frac{2}{\sqrt{\pi}}$ $P\left(\frac{Z_1}{\lvert Z_2 \rvert} < t\right) = \frac{1}{\pi(t^2+1)}$ Cauchy's

Distribution	Mean	Variance	MGF	$E(X^k)$	Relation to other distribution	
Exponential (λ) $f_T(t) = \lambda e^{-\lambda t}$ $F_T(t) = 1 - e^{-\lambda t}$ $P(T>t) = 1 - F_T(t)$ $= e^{-\lambda t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$ $= \frac{1}{1 - t/\lambda}$ $= \sum_{n=0}^{\infty} \left(\frac{t}{\lambda}\right)^n$	$\frac{n!}{\lambda^n}$	Memoryless. $P(T>t) = G(s)$ $G(s+t) = G(s)\,G(t)$ $G(nt) = G(t)^n$ $G(xt) = G(t)^x$	$Y \sim Exp(\lambda)$ $X = \lambda Y \sim Exp(1)$ $(X - t \mid X>t) \sim Exp(\lambda)$ $E(X \mid X>t) = t + \frac{1}{\lambda}$ $E(X - t \mid X>t) = 1/\lambda$

11. Chicken & Egg Problem

$N \sim \text{Poisson}(\lambda) = \#\text{eggs}$

$X|N \sim \text{Bino}(N, p) = \#\text{eggs hatched given total } N \text{ eggs}$

$X = \#\text{eggs hatched}$

define $Y = N - X$

$X + Y = N$

$\Rightarrow$ Joint PMF of $X, Y = P(X,Y) = \sum_N P(X,Y|N) \cdot P(N) = P(X=i, Y=j | N=i+j) \cdot P(N=i+j)$

$= P(X=i | N=i+j) \cdot P(N=i+j) = {}^{i+j}C_i \, p^i (1-p)^{N-i} \cdot \frac{e^{-\lambda} \lambda^{i+j}}{(i+j)!}$

$$= \frac{(i+j)!}{i!\, j!} \frac{p^i (1-p)^j}{(i+j)!} e^{-\lambda(p+1-p)} \lambda^{i+j} = \boxed{\underbrace{\frac{(p\lambda)^i e^{-\lambda p}}{i!}}_{\text{Poisson}(\lambda p)} \cdot \underbrace{\frac{(1-p)^j \lambda^j e^{-\lambda(1-p)}}{j!}}_{\text{Poisson}(\lambda(1-p))} = P(X=i, Y=j)}$$